PMM U.S.S.R.,Vol.45,pp.122-126 0021-8928/82/1 0122 $7.50/0

UDC 539.3

RIGID ELLIPSOIDAL DISC AND NEEDLE IN AN ANISOTROPIC ELASTIC MEDIUM*

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The problem of stress concentration on an absolutely rigid ellipsoidal disc and needle in an arbitrary anisotropic elastic medium subjected to a homogeneous external field is solved. The ellipsoidal disc (needle) is understood to be an ellipsoidal inclusion one of whose dimensions is small (large) compared to the other two. Explicit expressions are obtained and studied, for the stresses on the whole surface of the stiff inclusion. It is shown that the stresses are singular (large compared to one, but finite) in the neighborhood of the needle endface for tension and shear acting in the plane of the edge. Explicit expressions are presented for the stress on the surface in the particular case of a transversally isotropic medium. In general solution of the problem concerning stress concentration on the surface of an ellipsoidal inhomogeneity /1/ and the method of expanding the solution in a small parameter /2/ based on application of the theory of generalized functions, special methods of extracting singularities and regularizing divergent integrals are used.

1. The stresses $\sigma^{\alpha\beta}(\mathbf{n})$ on the surface of an ellipsoidal inclusion in an external homogeneous field $\sigma_0^{\lambda\mu}$ have the form

$$\sigma^{\alpha\beta}(\mathbf{n}) = F^{\alpha\beta\cdot\cdot}_{\cdot\cdot\lambda\mu}(\mathbf{n})\,\sigma_0^{\lambda\mu} \tag{1.1}$$

Here $F(\mathbf{n})$ is the tensor stress concentration coefficient dependent on the normal $\mathbf{n} = \{n_1, n_2, n_3\}$ to the surface of an ellipsoid with the semiaxes a_α $(\alpha = 1, 2, 3)$, and representable as the product of two factors

$$E^{\alpha\beta\cdot\cdot}_{\cdot\cdot\lambda\mu}(\mathbf{n}) = B^{\alpha\beta\cdot\cdot}_{\cdot\cdot\sigma\tau}(\mathbf{n})\,R^{\sigma\tau\cdot\cdot}_{\cdot\cdot\lambda\mu} \tag{1.2}$$

The first factor $B(\mathbf{n})$ depends on the ellipsoid parameters implicitly through the normal $\mathbf{n}$ and upon passing to the limit cases contains no singularities. For a rigid inclusion

$$B^{\alpha\beta\cdot\cdot}_{\cdot\cdot\sigma\tau}(\mathbf{n}) = c^{\alpha\beta\varkappa\rho}K_{\varkappa\rho\sigma\tau}(\mathbf{n})$$

where $c^{\alpha\beta\varkappa\rho}$ is the tensor of the elastic constants of the external medium, and the tensor $K(\mathbf{n})$ is constructed explicitly in terms of the Fourier transform of the Green's tensor for an arbitrary anisotropic medium. The expression for the tensor $K(\mathbf{n})$ for an isotropic medium is presented in /1/ in an $\{x^1, x^2, x^3\}$ coordinate system coupled rigidly to the ellipsoid, and the components of $K(\mathbf{n})$ for an orthotropic medium have the form

$$K_{1111}(\mathbf{n}) = \frac{n_1^2}{L}\{c_{55}c_{66}n_1^4 + c_{44}(c_{22}n_2^4 + c_{33}n_3^4) + n_2^2n_3^2(\Delta_{11} - 2c_{23}c_{44}) + n_1^4[n_2^2(c_{22}c_{55} + c_{44}c_{66}) + n_3^2(c_{33}c_{66} + c_{44}c_{66})]\}$$

$$K_{1122}(\mathbf{n}) = \frac{n_1^2n_2^2}{L}\{n_3^2[(c_{13} + c_{55})(c_{23} + c_{44}) - c_{33}(c_{12} + c_{66})] - (c_{12} + c_{66})(c_{55}n_1^2 + c_{44}n_2^2)\}$$

$$K_{1212}(\mathbf{n}) = \frac{1}{L}\{c_{33}n_3^4(c_{55}n_1^2 + c_{44}n_2^2) + n_1^4[n_2^2(c_{11}c_{44} - 2c_{12}c_{55}) + n_3^2(\Delta_{22} - 2c_{13}c_{55})] + n_2^4[n_1^2(c_{22}c_{55} - 2c_{12}c_{44}) + n_3^2(\Delta_{11} - 2c_{23}c_{44})] + 2n_1^2n_2^2n_3^2[\Delta_{12} + c_{23}c_{55} + c_{13}c_{44} + 2c_{44}c_{55} + c_{11}c_{55}n_1^6 + c_{22}c_{44}n_2^6\}$$

$$K_{1213}(\mathbf{n}) = \frac{n_2n_3}{L}\{n_2^2n_3^2(\Delta_{11} - 2c_{23}c_{44}) + n_1^2(n_2^2\Delta_{13} + n_3^2\Delta_{12}) + c_{44}(c_{22}n_2^4 + c_{33}n_3^4) + n_1^4(\Delta_{23} - c_{11}c_{44})\}$$

$$K_{1211}(\mathbf{n}) = \frac{n_1n_2}{2L}\{c_{44}(c_{22}n_2^4 + c_{33}n_3^4) - c_{12}c_{55}n_1^4 + n_1^2n_3^2(\Delta_{12} + c_{55}c_{23} + c_{44}c_{13} + 2c_{44}c_{55}) + n_1^2n_2^2(c_{22}c_{55} - c_{12}c_{44}) + n_2^2n_3^2(\Delta_{11} - 2c_{23}c_{44})\}$$

$$K_{1233}(\mathbf{n}) = \frac{n_1^2n_2^2n_3^2}{2L}\{n_1^2(\Delta_{23} + c_{12}c_{55} - c_{11}c_{44}) + n_2^2(\Delta_{12} + c_{12}c_{44} - c_{22}c_{55}) - n_3^2(c_{13}c_{44} + c_{23}c_{56} + 2c_{44}c_{55})\}$$

$$L = c_{11}c_{55}c_{66}n_1^6 + c_{22}c_{44}c_{66}n_2^6 + c_{33}c_{44}c_{55}n_3^6 + n_1^4\{n_2^2[c_{55}\Delta_{33} + c_{66}(c_{11}c_{44} - 2c_{12}c_{55}) + n_3^2[c_{66}\Delta_{22} + c_{55}(c_{11}c_{44} - 2c_{13}c_{66})\} + n_2^4\{n_1^2[c_{44}\Delta_{33} + c_{66}(c_{22}c_{55} - 2c_{12}c_{55})] + n_3^2[c_{66}\Delta_{11} + c_{44}(c_{22}c_{55} - 2c_{23}c_{66})]\} + n_3^4\{n_2^2[c_{55}\Delta_{11} + c_{44}(c_{33}c_{66} - 2c_{23}c_{55})] + n_1^2(c_{44}\Delta_{22} + c_{55}(c_{33}c_{66} - 2c_{13}c_{44})]\} + n_1^2n_2^2n_3^2[\Delta + 2c_{44}c_{55}c_{66} + 2c_{44}(\Delta_{23}c_{13}c_{66}) + 2c_{55}(\Delta_{13} + c_{12}c_{44}) + 2c_{66}(\Delta_{12} + c_{23}c_{55})]$$

$$c_{\alpha\beta} = c^{\alpha\alpha\beta\beta}\ (\alpha, \beta = 1, 2, 3),\ c_{44} = c^{2323},\ c_{55} = c^{1313},\ c_{66} = c^{1212}$$

*Prikl.Matem.Mekhan.,45,No.1,165-170,1981

Here Δ and $\Delta_{\alpha\beta}$ are, respectively, the determinant and cofactor of the element $c_{\alpha\beta}$ of the matrix $\|c_{\alpha\beta}\|$ $(\alpha, \beta = 1, 2, 3)$. The remaining components of the tensor $K_{\alpha\beta\lambda\mu}(\mathbf{n})$ are obtained from a cyclic permutation of the subscripts 1,2,3 and 4,5,6.

The second factor R in (1.2) depends substantially on the shape of the inclusion. For an ellipsoid this is a constant tensor of orthorhombic structure, inverse to the tensor

$$B = \frac{a_1 a_2 a_3}{4\pi} \int\limits_{\Omega} B(\mathbf{n}) \frac{dn}{(\mathbf{n}a^2\mathbf{n})^{3/2}}$$

$$\mathbf{n}a^2\mathbf{n} = a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2$$

The tensor R becomes singular upon going over to the disc and needle.

To describe the geometry of the limit case being considered, we introduce, as in /2/, the dimensionless parameters

$$\alpha = a_2/a_1, \ \xi = a_3/a_2 \ (a_1 \geqslant a_2 \geqslant a_3)$$

where $\alpha \ll 1$, $\xi \sim 1$ corresponds to the needle $\xi \ll 1$, $\alpha \sim 1$ to a disc, and $\alpha \ll 1$, $\xi \ll 1$ to an elongated disc.

The solution of the stress concentration problem at a needle and disc reduces to the calculation of the principal terms in the expansion of the tensor R in an appropriate small parameter. The solution is sought in the class of generalized functions.

2. Let us first consider a stiff disc $(\xi \ll 1, \alpha \sim 1)$. Applying the method of expanding the tensor B in a small parameter ξ proposed in /2/, we obtain

$$B = B_0 + \xi B_1 + O(\xi^2), \quad B_0 = \frac{\alpha}{2\pi} \int\limits_0^{2\pi} \frac{B(\varphi, \pm 1)}{\cos^2\varphi + \alpha^2 \sin^2\varphi} d\varphi$$

$$B_1 = \frac{\alpha^2}{4\pi} \int\limits_0^{2\pi} \frac{d\varphi}{(\cos^2\varphi + \alpha^2\sin^2\varphi)^{3/2}} \int\limits_{-1}^{1} \frac{B(\varphi, t) - B(\varphi, \pm 1)}{(1 - t^2)^{3/2}} dt$$

$$B(\varphi, t) \equiv B(\mathbf{n}(\varphi, t)), \quad t = \cos\theta$$

(φ, θ are specific coordinates with a polar axis directed along the x^3 axis).

We note that the tensor B_0 is easily evaluated for an arbitrary anisotropic medium, and equals the value of the tensor $B(\mathbf{n})$ for $n_1 = n_2 = 0$, $n_3 = 1$. In particular, the components of B_0 different from zero for an orthotropic medium have the form

$$B_{0\cdot\cdot 33}^{11\cdot\cdot} = \frac{c_{13}}{c_{33}}, \quad B_{0\cdot\cdot 33}^{22\cdot\cdot} = \frac{c_{23}}{c_{33}}, \quad B_{0\cdot\cdot 33}^{33\cdot\cdot} = 1, \quad B_{0\cdot\cdot 13}^{13\cdot\cdot} = B_{0\cdot\cdot 23}^{23\cdot\cdot} = \frac{1}{2}$$

Without performing specific calculations of the coefficient B_1, this permits setting up orders of the singularities in the concentration coefficient components for an arbitrary anisotropic medium, and the stresses on a disc surface, respectively. In fact, the components of the tensor B have the following orders in ξ:

$$B_{\cdot\cdot\beta\beta}^{\alpha\alpha\cdot\cdot} \sim B_{\cdot\cdot 12}^{12\cdot\cdot} \sim \xi, \quad B_{\cdot\cdot 33}^{\alpha\alpha\cdot\cdot} \sim B_{\cdot\cdot 13}^{13\cdot\cdot} \sim B_{\cdot\cdot 23}^{23\cdot\cdot} \sim \mathrm{const} + O(\xi) \quad (\alpha = 1, 2, 3; \ \beta = 1, 2)$$

The components of the inverse tensor to R have singularities of the order of ξ^{-1} in the components $R_{\cdot\cdot\alpha\alpha}^{11\cdot\cdot}, R_{\cdot\cdot\alpha\alpha}^{22\cdot\cdot}, R_{\cdot\cdot 12}^{12\cdot\cdot}$.

Taking account of only singular terms, from (1.1) and (1.2) we obtain an expression for the stress on a disc surface

$$\sigma^{\alpha\beta}(\mathbf{n}) = [B_{\cdot\cdot 11}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot\lambda\lambda}^{11\cdot\cdot} + B_{\cdot\cdot 22}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot\lambda\lambda}^{22\cdot\cdot}]\sigma_0^{\lambda\lambda} + 4B_{\cdot\cdot 12}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot 12}^{12\cdot\cdot}\sigma_0^{12}$$

It is hence seen that singularities in the stresses can appear subjected to external tension $\sigma_0^{\lambda\lambda}$ $(\lambda = 1, 2, 3)$ and shear σ_0^{12} acting in the plane of the disc edge $(n_3 = 0)$

$$\sigma^{\alpha\beta}(\mathbf{n}) = [B_{\cdot\cdot 11}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot\lambda\lambda}^{11\cdot\cdot} + B_{\cdot\cdot 22}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot\lambda\lambda}^{22\cdot\cdot}]\sigma_0^{\lambda\lambda}, \quad \sigma^{\alpha\beta}(\mathbf{n}) = 4B_{\cdot\cdot 12}^{\alpha\beta\cdot\cdot}(\mathbf{n}) R_{\cdot\cdot 12}^{12\cdot\cdot}\sigma_0^{12}$$

Taking into account the structure of the tensor $B(\mathbf{n})$, we obtain that the behavior of the singular components of the stresses is governed by the following quantities:

For tension

$$\sigma^{\alpha\alpha}(\mathbf{n}) \sim [n_1^2 R_{\cdot\cdot\lambda\lambda}^{11\cdot\cdot} + n_2^2 R_{\cdot\cdot\lambda\lambda}^{22\cdot\cdot}]\sigma_0^{\lambda\lambda} \sim (n_1^2 + n_2^2)\xi^{-1}\sigma_0^{\lambda\lambda}, \quad \sigma^{\alpha\beta}(\mathbf{n}) \sim n_\alpha n_\beta [R_{\cdot\cdot\lambda\lambda}^{11\cdot\cdot} + R_{\cdot\cdot\lambda\lambda}^{22\cdot\cdot}]\sigma_0^{\lambda\lambda} \sim n_\alpha n_\beta \xi^{-1}\sigma_0^{\lambda\lambda}, \ \alpha \neq \beta$$

For shear

$$\sigma^{\alpha\alpha}(\mathbf{n}) \sim n_1 n_2 \xi^{-1}\sigma_0^{12}, \quad \sigma^{12}(\mathbf{n}) \sim (n_1^2 + n_2^2)\,\xi^{-1}\sigma_0^{12}, \quad \sigma^{13}(\mathbf{n}) \sim n_1 n_3 \xi^{-1}\sigma_0^{12}, \quad \sigma^{23}(\mathbf{n}) \sim n_2 n_3 \xi^{-1}\sigma_0^{12}$$

It follows from the formulas obtained that singularities in the stresses $\sigma^{\alpha\alpha}(\mathbf{n})$ $(\alpha = 1, 2, 3)$ and $\sigma^{12}(\mathbf{n})$ occur at the edge of the disc $(n_1^2 + n_2^2 = 1)$ and in its neighborhood, while the stress surge phenomenon holds for the shear σ_0^{12} in stresses $\sigma^{13}(\mathbf{n})$ and $\sigma^{23}(\mathbf{n})$ (the stresses reach a maximum in the neighborhood of the edge although they are zero on the edge itself).

3. Let us consider a needle $(\alpha \ll 1,\ \xi \sim 1)$. Performing transformations in the tensor B, analogous to transformations for a hollow needle /2/ and using the method of regularizing integrals of the generalized homogeneous functions (see /3/, p.385), we obtain the following expansion for the tensor B:

$$B = B_0 + B_1\alpha^2 \ln\alpha + O(\alpha^2) \tag{3.1}$$

$$B_0 = \frac{\xi}{2\pi}\int_0^{2\pi} \frac{B(\varphi, 0)}{\cos^2\varphi + \xi^2\sin^2\varphi}\, d, \quad \varphi B_1 = -\int_0^{2\pi} B_{t}''(\varphi, 0)\, d\varphi$$

$$B(\varphi, 0) \equiv B(\mathbf{n}(\varphi, t))|_{t=0}, \quad t = \cos\theta$$

Here φ, θ are spherical coordinates with a polar axis directed along the axis x^1.

In constrast to a hollow needle, in order to calculate the principal term in the expansion of the tensor R it is necessary to take account of at least the first two terms in B since the tensor B_0 has no inverse.

Using the expansion (3.1), we establish that the components $B^{\beta\beta\cdot\cdot}_{\cdot\cdot 11}$ of the tensor B are of the orders $\alpha^2\ln\alpha$, and all the rest the orders $[\text{const} + O(\alpha^2\ln\alpha)]$. Correspondingly, a singularity of order $(\alpha^2\ln\alpha)^{-1}$ holds in the tensor R in the components $R^{11\cdot\cdot}_{\cdot\cdot\beta\beta}$ $(\beta = 1, 2, 3)$.

From (1.1) and (1.2) we obtain the singular stress components on the needle surface

$$\sigma^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta\cdot\cdot}_{\cdot\cdot 11}(\mathbf{n})\, R^{11\cdot\cdot}_{\cdot\cdot\lambda\lambda}\sigma_0^{\lambda\lambda} \quad (\lambda = 1,\ 2,\ 3)$$

It is hence seen that singular stresses occur on the needle surface only under the effect of external tensile forces. Pure shear does not produce singularities in the stresses. Taking account of the structure of the tensor $B(\mathbf{n})$, it can be shown that the behavior of the stresses on the needle surface is determined by the following quantities

$$\sigma^{\beta\beta}(\mathbf{n}) \sim n_1^2(\alpha^2\ln\alpha)^{-1}\sigma_0^{\lambda\lambda} \quad (\beta = 1, 2, 3)$$

$$\sigma^{23}(\mathbf{n}) \sim n_1^2 n_2 n_3 (\alpha^2\ln\alpha)^{-1}\sigma_0^{\lambda\lambda}$$

$$\sigma^{12}(\mathbf{n}) \sim n_1 n_2 (\alpha^2\ln\alpha)^{-1}\sigma_0^{\lambda\lambda}, \quad \sigma^{13}(n) \sim n_1 n_3 (\alpha^2\ln\alpha)^{-1}\sigma_0^{\lambda\lambda}$$

Analyzing the expressions obtained, we find that the stresses $\sigma^{\beta\beta}(\mathbf{n})$ are singular at the needle endface $(n_1 = 1)$ and its neighborhood, while the stress surge phenomenon holds in the components $\sigma^{\alpha\beta}(\mathbf{n})$ $(\alpha \neq \beta)$ in the neighborhood of the endfaces.

4. Let us consider the case of an elongated disc $(\alpha \ll 1,\ \xi \ll 1)$ which is of independent interest. Expansions of the tensors B and R can be obtained for an elongated disc by passing to the limit from the formulas for a disc of finite size as $\alpha \to 0$, or from the formulas for a needle as $\xi \to 0$. Passing to the limit in (3.1), say, as $\xi \to 0$, we obtain

$$B = B_0 + B_1\xi + B_2\xi\alpha^2\ln\alpha + \ldots$$

$$B_0 = B(\varphi, t)|_{\varphi=\pi/2,\ t=0} = B\left(\frac{\pi}{2}, 0\right)$$

$$B_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{B(\varphi, 0) - B\left(\frac{\pi}{2}, 0\right)}{\cos^2\varphi}\, d\varphi, \quad B_2 = -\frac{1}{2\pi}\int_0^{2\pi} B_{t}''(\varphi, 0)\, d\varphi$$

Here the expresion for B_1 is written in regularized form. Continuing reasoning analogous to the preceding, we establish that the qualitative pattern of the stress behavior on the surface of an elongated disc is exactly as in the case of a finite size disc, i.e., the singularities in the stresses hold at the disc edge or in its neighborhood. However, if the orders of the singularities for shear forces are identical in both cases, then under tension stresses of the order of ξ^{-1} hold on the elongated disc only at side points of the edge, and grow to magnitudes on the order of $(\xi\alpha^2\ln\alpha)^{-1}$ in the neighborhood of the endfaces.

Let us note that in contrast to cavities /2/, the order of the singularities in the stresses on rigid inclusions depends substantially on the quantity of small parameters describing the geometry of the inclusion: singularities in a needle and disc are characterized by one small parameter, and by two in an elongated disc.

5. In conclusion, let us present explicit expressions for the stresses along the principal sections $n_1 = 0$, $n_2 = 0$ of the surface of the rigid inclusions in a transversely-isotropic medium. We write the stresses in a local coordinate system coupled to the normal n at each point of the surface as follows:

$$\mathbf{e}_{3'} = \mathbf{n}, \quad \mathbf{e}_{1'} = \frac{\mathbf{n} \times \mathbf{e}_3}{\sqrt{n_1^2 + n_2^2}}, \quad \mathbf{e}_{2'} = \frac{\mathbf{n} \times (\mathbf{n} \times \mathbf{e}_3)}{\sqrt{n_1^2 + n_2^2}}$$

($\mathbf{e}_\alpha$ are directions of the coordinate system x^α, and $\mathbf{e}_{\alpha'}$ are directions of the local system). Then the stress $\sigma^{3'3'}(\mathbf{n})$ is always along the normal to the surface, $\sigma^{1'1'}(\mathbf{n})$ perpendicular to the plane of the section, and $\sigma^{2'2'}(\mathbf{n})$ along the section contour.

Because of awkwardness of the general expressions, we present formulas for the stresses $\sigma^{\alpha'\beta'}(\mathbf{n})$ only with singular terms taken into account. In the section $n_2 = 0$ they have the following form for an orthotropic medium

$$\sigma^{1'1'}(\mathbf{n}) = \frac{n_1^2}{f(\mathbf{n})}\left[n_1^2 c_{13} c_{55} - n_3^2 (\Delta_{12} + c_{23} c_{55})\right] R^{11\cdot\cdot}_{\cdot\cdot\alpha\alpha} \sigma_0^{\alpha\alpha} \tag{5.1}$$

$$\sigma^{2'2'}(\mathbf{n}) = \frac{n_1^2}{f(\mathbf{n})}\left[n_1^4 c_{13} c_{55} + (\Delta_{22} - c_{55}(c_{13} + 2c_{33}))\, n_3^4 + n_1^2 n_3^2 c_{55} (c_{11} - c_{33} + 2c_{13})\right] R^{11\cdot\cdot}_{\cdot\cdot\alpha\alpha} \sigma_0^{\alpha\alpha}$$

$$\sigma^{3'3'}(\mathbf{n}) = n_1^2 R^{11\cdot\cdot}_{\cdot\cdot\alpha\alpha} \sigma_0^{\alpha\alpha} + 4 n_1 n_2 R^{12\cdot\cdot}_{\cdot\cdot12} \sigma_0^{12}, \quad \sigma^{1'2'}(\mathbf{n}) = \frac{2 n_1^2 n_3 (c_{44} - c_{66})}{c_{44} n_3^2 + c_{66} n_1^2} R^{12\cdot\cdot}_{\cdot\cdot12}$$

$$\sigma^{2'3'}(\mathbf{n}) = n_1 n_3 R^{11\cdot\cdot}_{\cdot\cdot\alpha\alpha} \sigma_0^{\alpha\alpha}, \quad \sigma^{1'3'}(\mathbf{n}) = -2 n_1 R^{12\cdot\cdot}_{\cdot\cdot12} \sigma_0^{12}$$

$$f(\mathbf{n}) = c_{55}(c_{11} n_1^4 + c_{33} n_3^4) + (\Delta_{22} - 2 c_{13} c_{55})\, n_1^2 n_3^2$$

To obtain the stresses $\sigma^{\alpha'\beta'}(\mathbf{n})$ in the section $n_1 = 0$ it is necessary to execute the change of subscript $1 \leftrightarrow 2$, $4 \leftrightarrow 5$ in the right sides of (5.1).

For a transversely-isotropic medium with an axis of elastic symmetry directed along the axis x^1, the components $c_{\alpha\beta}$ are

$$c_{11} = E_1 (1 - \nu_1^2)\Delta_1, \qquad c_{22} = c_{33} = E_1 (\rho - \nu_1^2)\Delta_1$$

$$c_{12} = c_{13} = \nu_2 E_1 (1 + \nu_1)\Delta_1, \quad c_{23} = E_1 (\nu_1 \rho + \nu_2^2)\Delta_1$$

$$c_{44} = {}^1/_2 (c_{22} - c_{33}), \qquad c_{55} = c_{66} = G$$

$$\rho = E_1 / E_2, \quad \Delta_1 = (1 + \nu_1)^{-1}[\rho (1 - \nu_1) - 2\nu_2^2]^{-1}$$

Here E_1, ν_1 and E_2, ν_2 are the Young's moduli and Poisson's ratios of the media in the plane of isotropy and the direction of the elastic symmetry axis, respectively, and G is the shear modulus in the direction of the axis of symmetry.

The components of the tensor R for the elongated disc have the form

$$R^{11\cdot\cdot}_{\cdot\cdot11} = \frac{1}{\xi}\frac{G}{E_2}\left\{\frac{1}{\alpha^2 |\ln \alpha|} + \frac{(1 - g)(1 - \nu_1 + g)(1 - \nu_1)}{4\varkappa(3 - \nu_1)}\right\} \tag{5.2}$$

$$R^{11\cdot\cdot}_{\cdot\cdot22} = -\frac{1}{\xi}\frac{G}{E_2}\left\{\frac{\nu_2}{\alpha^2 |\ln \alpha|} + \frac{(1 - \nu_1 + g)(\nu_2 - 4\varkappa)}{4\varkappa(3 - \nu_1)}\right\}$$

$$R^{11\cdot\cdot}_{\cdot\cdot33} = -\frac{1}{\xi}\frac{G}{E_1}\left\{\frac{\nu_2}{\alpha^2 |\ln \alpha|} + \frac{(1 - \nu_1 + g)(\nu_2 + 4\nu_1\varkappa)}{4\varkappa(3 - \nu_1)}\right\}$$

$$R^{22\cdot\cdot}_{\cdot\cdot11} = \frac{1}{\xi}\frac{G}{E_2}\frac{g(1 - \nu_1)}{3 - \nu_1}, \quad R^{22\cdot\cdot}_{\cdot\cdot22} = \frac{1}{\xi}\frac{G}{E_1}\frac{4\varkappa - \nu_2}{3 - \nu_1}$$

$$R^{22\cdot\cdot}_{\cdot\cdot33} = -\frac{1}{\xi}\frac{G}{E_1}\frac{4\nu_1\varkappa}{3 - \nu_1}, \quad R^{12\cdot\cdot}_{\cdot\cdot12} = \frac{1}{2\xi}$$

$$g = \frac{E_2 \nu_2}{G(1 + \nu_1)}, \quad \varkappa = \frac{E_1}{2G(1 + \nu_1)}$$

For the needle

$$R^{11\cdot\cdot}_{\cdot\cdot22} = R^{11\cdot\cdot}_{\cdot\cdot33} = -\nu_2, \quad R^{11\cdot\cdot}_{\cdot\cdot11} = -\frac{1}{\alpha^2 \ln \alpha}\frac{G\nu_2}{\xi E_1} \tag{5.3}$$

In the isotropic case $\sigma^{\alpha'\beta'}(\mathbf{n})$ are obtained from (5.1)–(5.3) for $\varkappa = 1$, $g = 2\nu$, $G/E_1 = G/E_2 = {}^1/_2(1+\nu)$ where ν is the Poisson's ratio.

In the axisymmetric problem ($\xi = 1$, $\sigma_0^{22} = \sigma_0^{33}$, $\sigma_0^{\alpha\beta} = 0$ $(\alpha \neq \beta)$) the expressions (5.1) and (5.3) agree with those known /4/.

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Translated by M.D.F.